

from <http://www.escape.com/~paulg53/math/pi/greg/index.html> Paul's Page of Pi

Paul's Page of Pi

Gregory, Leibniz, and Machin

Using the Arctangent Infinite Series

Two mathematicians, James Gregory (1638-1675) and Gottfried Wilhelm Leibniz (1646-1716), each working independantly of one another, discovered an infinite series for $\arctan x$ in the 1670's. This arctangent series led to the first ever found infinite series for π . Here is the arctangent series that they discovered:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

For clarity, it can be rewritten as:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n+1}}{2n+1} \right)$$

By expanding this series to an arbitrary number of terms we can approximate any arctangent to arbitrary precision. Gregory and Leibniz then found an equation for π in terms of the arctangent. Here is a derivation of that equation:

iterations of the series!

Machin's Variation

In 1706, John Machin introduced a variation on the Gregory/Leibniz series with a significantly increased rate of convergence, which made it a much more practical method of calculation. It's so effective that Machin's formula is one that is still used today by computer programmers to calculate the digits of π . I wrote a program in C++ which is based on Machin's formula, and I used it to calculate [the first 8192 digits](#) of π . Here is the derivation of Machin's formula. We start by defining β by:

$$\tan \beta = \frac{1}{5}$$

Then we use the double-angle formula for tan to find:

$$\tan 2\beta = \frac{2 \tan \beta}{1 - \tan^2 \beta} = \frac{5}{12}$$

Using the tan double-angle formula again we learn:

$$\tan 4\beta = \frac{2 \tan 2\beta}{1 - \tan^2 2\beta} = \frac{120}{119}$$

We note that this number, 120/119, is only 1/119 off from 1, which has an arctan of $\pi/4$. Machin decided to calculate this difference (120/119 - 1) in terms of the angles:

$$\tan \left(4\beta - \frac{\pi}{4} \right) = \frac{\tan 4\beta - 1}{1 + \tan 4\beta} = \frac{1}{239}$$

Therefore:

$$\arctan \left(\frac{1}{239} \right) = 4\beta - \frac{\pi}{4}$$

Now, we can substitute β for its arctangent value:

$$\arctan \left(\frac{1}{239} \right) = 4 \arctan \left(\frac{1}{5} \right) - \frac{\pi}{4}$$

Finally, by solving for $\pi/4$ we get:

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

Using his own formula, Machin was able to calculate 100 places of π in 1706.

Since Machin's time many many other arctangent formulae for pi have been discovered. The formulae are, like Machin's formula, just the sum or difference of a few arctangent values.

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Paul's Page of Pi: Paul's 8192 Digits of Pi

Paul's 8192 Digits of Pi

I calculated these digits using a C++ program I wrote that uses Machin's formula. The code is based on a C program by Roy Williams. The last few digits may be a little off. The calculations were done on a 66Mhz. PowerPC 601 chip.

3.

1415 9265 3589 7932 3846 2643 3832 7950 2884 1971 6939 9375
1058 2097 4944 5923 0781 6406 2862 0899 8628 0348 2534 2117
0679 8214 8086 5132 8230 6647 0938 4460 9550 5822 3172 5359
4081 2848 1117 4502 8410 2701 9385 2110 5559 6446 2294 8954
9303 8196 4428 8109 7566 5933 4461 2847 5648 2337 8678 3165
2712 0190 9145 6485 6692 3460 3486 1045 4326 6482 1339 3607

2602	4914	1273	7245	8700	6606	3155	8817	4881	5209	2096	2829
2540	9171	5364	3678	9259	0360	0113	3053	0548	8204	6652	1384
1469	5194	1511	6094	3305	7270	3657	5959	1953	0921	8611	7381
9326	1179	3105	1185	4807	4462	3799	6274	9567	3518	8575	2724
8912	2793	8183	0119	4912	9833	6733	6244	0656	6430	8602	1394
9463	9522	4737	1907	0217	9860	9437	0277	0539	2171	7629	3176
7523	8467	4818	4676	6940	5132	0005	6812	7145	2635	6082	7785
7713	4275	7789	6091	7363	7178	7214	6844	0901	2249	5343	0146
5495	8537	1050	7922	7968	9258	9235	4201	9956	1121	2902	1960
8640	3441	8159	8136	2977	4771	3099	6051	8707	2113	4999	9998
3729	7804	9951	0597	3173	2816	0963	1859	5024	4594	5534	6908
3026	4252	2308	2533	4468	5035	2619	3118	8171	0100	0313	7838
7528	8658	7533	2083	8142	0617	1776	6914	7303	5982	5349	0428
7554	6873	1159	5628	6388	2353	7875	9375	1957	7818	5778	0532
1712	2680	6613	0019	2787	6611	1959	0921	6420	1989	3809	5257
2010	6548	5863	2788	6593	6153	3818	2796	8230	3019	5203	5301
8529	6899	5773	6225	9941	3891	2497	2177	5283	4791	3151	5574
8572	4245	4150	6959	5082	9533	1168	6172	7855	8890	7509	8381
7546	3746	4939	3192	5506	0400	9277	0167	1139	0098	4882	4012
8583	6160	3563	7076	6010	4710	1819	4295	5596	1989	4676	7837
4494	4825	5379	7747	2684	7104	0475	3464	6208	0466	8425	9069
4912	9331	3677	0289	8915	2104	7521	6205	6966	0240	5803	8150
1935	1125	3382	4300	3558	7640	2474	9647	3263	9141	9927	2604
2699	2279	6782	3547	8163	6009	3417	2164	1219	9245	8631	5030
2861	8297	4555	7067	4983	8505	4945	8858	6926	9956	9092	7210
7975	0930	2955	3211	6534	4987	2027	5596	0236	4806	6549	9119
8818	3479	7753	5663	6980	7426	5425	2786	2551	8184	1757	4672
8909	7777	2793	8000	8164	7060	0161	4524	9192	1732	1721	4772
3501	4144	1973	5685	4816	1361	1573	5255	2133	4757	4184	9468
4385	2332	3907	3941	4333	4547	7624	1686	2518	9835	6948	5562
0992	1922	2184	2725	5025	4256	8876	7179	0494	6016	5346	6804
9886	2723	2791	7860	8578	4383	8279	6797	6681	4541	0095	3883
7863	6095	0680	0642	2512	5205	1173	9298	4896	0841	2848	8626
9456	0424	1965	2850	2221	0661	1863	0674	4278	6220	3919	4945
0471	2371	3786	9609	5636	4371	9172	8746	7764	6575	7396	2413
8908	6583	2645	9958	1339	0478	0275	9009	9465	7640	7895	1269
4683	9835	2595	7098	2582	2620	5224	8940	7726	7194	7826	8482
6014	7699	0902	6401	3639	4437	4553	0506	8203	4962	5245	1749
3996	5143	1429	8091	9065	9250	9372	2169	6461	5157	0985	8387
4105	9788	5959	7729	7549	8930	1617	5392	8468	1382	6868	3868
9427	7415	5991	8559	2524	5953	9594	3104	9972	5246	8084	5987
2736	4469	5848	6538	3673	6222	6260	9912	4608	0512	4388	4390
4512	4413	6549	7627	8079	7715	6914	3599	7700	1296	1608	9441

6948	6855	5848	4063	5342	2072	2258	2848	8648	1584	5602	8506
0168	4273	9452	2674	6767	8895	2521	3852	2549	9546	6672	7823
9864	5659	6116	3548	8623	0577	4564	9803	5593	6345	6817	4324
1125	1507	6069	4794	5109	6596	0940	2522	8879	7108	9314	5669
1368	6722	8748	9405	6010	1503	3086	1792	8680	9208	7476	0917
8249	3858	9009	7149	0967	5985	2613	6554	9781	8931	2978	4821
6829	9894	8722	6588	0485	7564	0142	7047	7555	1323	7964	1451
5237	4623	4364	5428	5844	4795	2658	6782	1051	1413	5473	5739
5231	1342	7166	1021	3596	9536	2314	4295	2484	9371	8711	0145
7654	0359	0279	9344	0374	2007	3105	7853	9062	1983	8744	7808
4784	8968	3321	4457	1386	8751	9435	0643	0218	4531	9104	8481
0053	7061	4680	6749	1927	8191	1979	3995	2061	4196	6342	8754
4406	4374	5123	7181	9217	9998	3910	1591	9561	8146	7514	2691
2397	4894	0907	1864	9423	1961	5679	4520	8095	1465	5022	5231
6038	8193	0142	0937	6213	7855	9566	3893	7787	0830	3906	9792
0773	4672	2182	5625	9966	1501	4215	0306	8038	4477	3454	9202
6054	1466	5925	2014	9744	2850	7325	1866	6002	1324	3408	8190
7104	8633	1734	6496	5145	3905	7962	6856	1005	5081	0665	8796
9981	6357	4736	3840	5257	1459	1028	9706	4140	1109	7120	6280
4390	3975	9515	6771	5770	0420	3378	6993	6007	2305	5876	3176
3594	2187	3125	1471	2053	2928	1918	2618	6125	8673	2157	9198
4148	4882	9164	4706	0957	5270	6957	2209	1756	7116	7229	1098
1690	9152	8017	3506	7127	4858	3222	8718	3520	9353	9657	2512
1083	5791	5136	9882	0914	4421	0067	5103	3467	1103	1412	6711
1369	9086	5851	6398	3150	1970	1651	5116	8517	1437	6576	1835
1556	5088	4909	9898	5998	2387	3455	2833	1635	5076	4791	8535
8932	2618	5489	6321	3293	3089	8570	6420	4675	2590	7091	5481
4165	4985	9461	6371	8027	0981	9943	0992	4488	9575	7128	2890
5923	2332	6097	2997	1208	4433	5732	6548	9382	3911	9325	9746
3667	3058	3604	1428	1388	3032	0382	4903	7589	8524	3744	1702
9132	7656	1809	3773	4440	3070	7469	2112	0191	3020	3303	8019
7621	1011	0044	9293	2151	6084	2444	8596	3766	9838	9522	8684
7831	2355	2658	2131	4495	7685	7262	4334	4189	3039	6864	2624
3410	7732	2697	8028	0731	8915	4411	0104	4682	3252	7162	0105
2652	2721	1166	0396	6655	7309	2547	1105	5785	3763	4668	2065
3109	8965	2691	8620	5647	6931	2570	5863	5662	0185	5810	0729
3606	5987	6486	1179	1045	3348	8503	4611	3657	6867	5324	9441
6680	3962	6579	7877	1855	6084	5529	6541	2665	4085	3061	4344
4318	5867	6975	1456	6140	6800	7002	3787	7659	1344	0171	2749
4704	2056	2230	5389	9456	1314	0711	2700	0407	8547	3326	9939
0814	5466	4645	8807	9727	0826	6830	6343	2858	7856	9830	5235
8089	3306	5757	4067	9545	7163	7752	5420	2114	9557	6158	1400
2501	2622	8594	1302	1647	1550	9792	5923	0990	7965	4737	6125

5176	5675	1357	5178	2966	6454	7791	7450	1129	9614	8903	0463
9947	1329	6210	7340	4375	1895	7359	6145	8901	9389	7131	1179
0429	7828	5647	5032	0319	8691	5140	2870	8085	9904	8010	9412
1472	2131	7947	6477	7262	2414	2548	5454	0332	1571	8530	6142
2881	3758	5043	0633	2175	1829	7986	6223	7172	1591	6077	1669
2547	4873	8986	6549	4945	0114	6540	6284	3366	3937	9003	9769
2656	7214	6385	3067	3609	6571	2091	8076	3832	7166	4162	7488
8800	7869	2560	2902	2847	2104	0317	2118	6082	0419	0004	2296
6171	1963	7792	1337	5751	1495	9501	5660	4963	1862	9472	6547
3642	5230	8177	0367	5159	0673	5023	5072	8354	0567	0403	8674
3513	6222	2477	1589	1504	9530	9844	4893	3309	6340	8780	7693
2599	3978	0541	9341	4473	7744	1842	6312	9860	8099	8886	8741
3260	4721	5695	1623	9658	6457	3021	6315	9819	3195	1673	5381
2974	1677	2947	8672	4229	2465	4366	8009	8067	6928	2382	8068
9964	0048	2435	4037	0141	6314	9658	9794	0924	3237	8969	0706
9779	4223	6250	8221	6889	5738	3798	6230	0159	3776	4716	5122
8935	7860	1588	1617	5578	2973	5233	4460	4281	5126	2720	3734
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3418	9948	5444	7345	6738	3162	4993	4191	3181	4809	2777	7103
8638	7734	3177	2075	4565	4532	2077	7092	1201	9051	6609	6280
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2735	6763	0354	4776	2803	5045	0777	2355	4710	5859	5487	0279
0814	3562	4014	5171	8062	4643	6267	9456	1275	3181	3407	8330
3362	5423	2783	9449	7538	2437	2058	3531	1477	1199	2606	3813
3467	7687	9695	9703	0983	3913	0771	0987	0408	5913	3746	4144
2822	7726	3465	9470	4745	8784	7787	2019	2771	5280	7317	6790
7707	1572	1344	4730	6057	0073	3492	4369	3113	8350	4931	6312
8404	2512	1925	6517	9806	9411	3528	0131	4701	3047	8164	3788
5185	2909	2854	5201	1658	3934	1965	6213	4914	3415	9562	5865
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3163	0577	7756	0688	8764	4624	8246	8579	2603	9535	2773	4803
0480	2900	5876	0758	2510	4747	0916	4396	1362	6760	4492	5627
4204	2083	2085	6611	9062	5454	3372	1315	3595	8450	6877	2460
2901	6187	6679	5240	6163	4252	2577	1954	2916	2991	9306	4553
7799	1403	7340	4328	7526	2888	9639	9587	9475	7291	7464	2635
7455	2540	7909	1451	3571	1136	9410	9119	3932	5191	0760	2082
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1697	1737	2784	7684	7268	6084	9003	3770	2424	2916	5130	0500
5168	3233	6435	0389	5170	2989	3922	3345	1722	0138	1280	6965
0117	8440	8745	1960	1212	2859	9371	6231	3017	1144	4846	4090
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7866	8088	1833	8510	2283	3450	8504	8608	2503	9302	1332	1971
5518	4306	3545	5007	6682	8294	9304	1377	6552	7939	7517	5461

3953	9846	8339	3638	3047	4611	9966	5385	8153	8420	5685	3386
2186	7252	3340	2830	8711	2328	2789	2125	0771	2629	4632	2956
3989	8989	3582	1167	4562	7010	2183	5646	2201	3496	7151	8819
0973	0381	1980	0497	3407	2396	1036	8540	6643	1939	5097	9019
0699	6395	5245	3005	4505	8068	5501	9567	3022	9219	1393	3918
5680	3449	0398	2059	5510	0226	3535	3619	2041	9947	4553	8593
8102	3439	5544	9597	7837	7902	3742	1617	2711	1723	6434	3543
9478	2218	1852	8624	0851	4006	6604	4332	5888	5698	6705	4315
4706	9657	4745	8550	3323	2334	2107	3015	4594	0516	5537	9068
6627	3337	9958	5115	6257	8432	2988	2737	2319	8987	5714	1595
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1268	0492	5648	7985	5614	5372	3478	6733	0390	4688	3834	3634
6553	7949	8641	9270	5638	7293	1748	7233	2083	7601	1230	2991
1367	9386	2708	9438	7993	6201	6295	1541	3371	4248	9283	0722
0126	9014	7546	6847	6535	7616	4773	7946	7520	0490	7571	5552
7819	6536	2132	3926	4061	6013	6358	1559	0742	2020	2031	8727
7605	2772	1900	5561	4842	5551	8792	5303	4351	3984	4253	2234
1576	2336	1064	2506	3904	9750	0865	6271	0953	5919	4658	9751
4131	0348	2276	9306	2474	3536	3256	9160	7815	4781	8115	2843
6679	5706	1108	6153	3150	4452	1274	7392	4544	9454	2368	2886
0613	4084	1486	3776	7009	6120	7151	2491	4043	0272	5386	0764
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0326	5918	4044	3780	5133	3894	5257	4239	9508	2965	9122	8508
5558	2157	2503	1071	2570	1266	8302	4029	2952	5220	1187	2676
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0783	8690	9291	6030	2884	0026	9104	1407	9288	6215	0784	2451
6709	0870	0069	9282	1206	6041	8371	8065	3556	7252	5325	6753
2861	2910	4248	7761	8258	2976	5157	9598	4703	5622	2629	3486
0034	1587	2298	0534	9896	5022	6291	7487	8820	2734	2092	2224
5339	8562	6476	6914	9055	6284	2503	9127	5771	0284	0279	9806
6365	8254	8892	6488	0254	5661	0172	9670	2664	0765	5904	2909
9456	8150	6526	5305	3718	2941	2703	3693	1378	5178	6090	4070
8667	1149	6558	3434	3476	9338	5781	7113	8645	5873	6781	2301
4587	6871	2660	3489	1390	9561	9728					

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